PMM U.S.S.R.,Vol.45,pp.94-100 0021-8928/82/1 0094 $7.50/0

UDC 530.182

ANALYSIS OF STEADY SOLUTIONS OF INVARIANT QUASI-LINEAR THIRD ORDER EQUATIONS*

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Singularities of solutions with steady structure of divergent quasi-linear third order equations proposed in /1/ are investigated. These equations written for functions of two independent variables, viz. space coordinate and time, simulate physical processes with dissipation and nonlinear and oscillatory effects. They define, in particular, unsteady potential motions in a layer of incompressible fluid with a free surface, for instance, unsteady plane jets or surface waves on shallow water. Another domain of their application is the investigation of properties of certain difference schemes of second order approximation in which similar equations are obtained as respective differential approximations of investigated schemes.

Steady solutions of such equations are qualitatively analyzed below in the phase plane. Interrelation between the properties of invariance and symmetry of solutions is established, and the possibility of passing to the limit of respective generalized solutions of the quasi-linear first order equation is investigated. Cases in which steady solutions exist in a moving coordinate system, which define the shock wave front and, also, steady solutions of the solitone (solitary wave) type are indicated. Numerical results of integration of unsteady input equations that confirm the results of qualitative analysis are presented for some of the investigated variants of steady solutions.

1. Consider the divergent invariant quasi-linear equations of the following type:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial t\partial x^2}+\beta\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial x^2}\right)+\varepsilon\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2=\alpha\frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

when $\alpha=\text{const}\geqslant 0$, $\varepsilon=\text{const}\geqslant 0$, and $\beta=\text{const}$ are of arbitrary sign. This equation can be taken as some third order regularizer for the quasi-linear first order equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0 \tag{1.2}$$

or for the Burgers equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\alpha\frac{\partial^2 u}{\partial x^2} \tag{1.3}$$

which maintains the fully determined invariance properties of these equations. All these equations are invariant to the geometric group of Poincaré transforms /1/.

The physical process described by Eq.(1.1) is characterized by the effects of nonlinearity of uu_x which also appear in (1.2) and (1.3), of dissipation of αu_{xx} appearing in (1.3), of "quadratic viscosity" $\varepsilon(u_x^2)_x$, oscillatory effects related to the presence of the term βu_{txx} when $\beta>0$, and to some effects inherent of the widely used Korteweg–de Vries Equation/2/ and of the regularized equation for long waves (terms $\beta(uu_{xx})_x$ and βu_{txx} for $\beta<0$).

Let us show that the invariant third order equation of form (1.1) defines unsteady potential motion in a layer of incompressible inviscid fluid with free surface.

Consider the layer of inviscid incompressible fluid of finite thickness and bounded on one side by the plane (x_1, x_2) which is either a wall or a plane of symmetry, and from the other by the free surface $y=h(\mathbf{x},t)$ with the y-axis directed vertically upward. We shall proceed from the following conventional system of equations and boundary conditions /4—6/: the Laplace equation

$$\nabla^2\varphi-\frac{1}{\beta}\varphi_{yy}=0\quad\left(\nabla=\mathbf{i}_1\frac{\partial}{\partial x_1}+\mathbf{i}_2\frac{\partial}{\partial x_2}\right) \tag{1.4}$$

the kinetic and dynamic boundary conditions at the free surface

*Prikl.Matem.Mekhan.,45,No.1,128-136,1981

$$h_t + (\nabla\varphi\nabla)h = \frac{1}{\beta}\varphi_y \tag{1.5}$$

$$\varphi_t + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2\beta}\varphi_y^2 + gh = 0 \tag{1.6}$$

and the boundary condition for the potential at $y = 0$

$$\varphi_y = 0 \tag{1.7}$$

where t is the time, $\mathbf{x} = (x_1, x_2)$ is the vector coordinate in the horizontal plane, $\mathbf{i}_1, \mathbf{i}_2$ are unit vectors of coordinate axes x_1, x_2, respectively, $\varphi(\mathbf{x}, y, t)$ is the potential of velocities, and g is the free fall acceleration.

All quantities in (1.4)—(1.7) are assumed dimensionless, normalized with the use of formulas

$$x_1 = x_1^\circ/l,\ x_2 = x_2^\circ/l,\ y = y^\circ/h_0,\ t = ct^\circ/l,\quad h = h^\circ/h_0,\ \varphi = \varphi^\circ/(cl),\ g = g^\circ h_0/c^2$$

where the small $^\circ$ superscript denotes dimensional parameters, l is a characteristic dimension of length relative to the x_1 and x_2 coordinates, h_0 is a characteristic dimension of length relative to the y-coordinate (e.g., the mean layer thickness), and c is the characteristic speed.

As in /4—6/, it is assumed below that the dimensionless geometric parameter $\beta = h_0^2/l^2$ is small. However unlike in /4—6/, we do not assume a priori the smallness of another parameter, the velocity ratio v_0/c, where v_0 is the characteristic velocity of particles on the fluid surface, and c is the characteristic rate of the process (e.g., in the case of waves on the surface of shallow water it is possible to take the wave propagation limit velocity $\sqrt{gh_0}$ for c).

Using (1.4)—(1.7) we obtain in the usual way /4—6/ the simplified equations for some specific case. Let us consider the two-dimensional motion of a fluid layer in the plane (x, y). Solution of the Laplace equation (1.4) that satisfies boundary condition (1.7) can be represented as an expansion in powers of y /5,6/

$$\varphi = f - \frac{\beta}{2}y^2 f_{xx} + \frac{\beta^2}{24}y^4 f_{xxxx} + O(\beta^3) \tag{1.8}$$

where function f depends only on variables x and t. Substituting (1.8) into the boundary conditions at the free surface (1.5) and (1.6) and introducing the horizontal velocity $u = f_x$ we obtain the system of equations

$$h_t + (uh)_x - \frac{\beta h^2}{2}\left[(hu_{xx})_x - \frac{2}{3}hu_{xxx}\right] = 0,\quad u_t + uu_x + gh_x - \frac{\beta}{2}[h^2(u_{tx} + uu_{xx} - u_x^2)]_x = 0 \tag{1.9}$$

accurate to $O(\beta^2)$. By rejecting the last terms containing parameter β as a multiplier, we obtain the equation of shallow water /4,5,7/ which defines one-dimensional wave propagation over a horizontal floor. Equations (1.9) represent the subsequent approximation which includes a dispersion correction for one-dimensional waves, and can be treated as some expanded variant of Boussinesq equations /4—6/.

Let the free surface $h(x, t)$ be subjected to slight perturbations. By substituting the mean value h_0 for the correction terms $h(x, t)$ which are proportional to β, we obtain instead of (1.9) a model system which in the absence of gravity can be successively solved. The second equation then assumes the form of an invariant quasi-linear equation of type (1.1) with $\alpha = 0$

$$u_t + uu_x - \beta_* u_{txx} - \beta_* uu_{xxx} + (\beta_*/2)(u_x^2)_x = 0 \quad (\beta_* = \beta h_0^2/2) \tag{1.10}$$

We stress that the last two terms that take into account nonlinearity of higher approximations have been retained in (1.10).

Equation (1.10) simulates one-dimensional unsteady motions in a layer of incompressible inviscid fluid in the absence of external mass forces. It can be used for defining the propagation of unsteady plane jets of perfect weightless fluid in space at constant pressure.

It should be pointed out that in the derivation of the Korteweg—de Vries equation, which defines a quasi-simple wave in the presence of dispersion, from the Boussinesq equations, only waves moving in the chosen direction are taken into consideration /4—6/. In the obtained third order equation of the form $u_t + uu_x + \beta u_{xxx} = 0$ the property of invariance relative direction change (simultaneous substitution of $-x$ for x and $-u$ for u) is violated.

Note that the third order equation similar to (1.1) appears in investigations of some difference scheme of second order approximations as a differential approximation of such schemes /8,9/, when they are used in the numerical integration of Eq.(1.2). Knowledge of such equations is useful in the study of causes of the appearance oscillations of numerical solutions in the domain of large gradients of considered functions.

This can be illustrated on the example of numerical solution of the simplest quasi-linear equation (1.2) using the different scheme of running calculations of second order approximation /10/. The respective differential approximation of the scheme (on condition that the integration step τ with respect to time is substantially smaller than the space coordinate step h) is of the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{h^2}{8}\frac{\partial^3 u}{\partial t\,\partial x^2}+\frac{h^2}{24}\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial x^2}\right)+\frac{h^2}{24}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2=0$$

accurate within $O(h^4)$, and coincides with the left-hand side of Eq.(1.1) except for the constant coefficients at higher derivatives.

2. Let Eq.(1.1) have steady solutions in the coordinate system moving at constant velocity U relative to the initial one.

We introduce the new coordinate $X = x - Ut$ and write down the equation that defines the steady solutions of Eq.(1.1) in the system of coordinates (X, t) which after a single integration with respect to X yields

$$\beta(u-U)\frac{d^2u}{dX^2}=\alpha\frac{du}{dX}-\varepsilon\left(\frac{du}{dX}\right)^2-\frac{u}{2}(u-2U)+C \tag{2.1}$$

where C is the arbitrary constant of integration. We set for simplicity $C = 0$ and introduce in (2.1) the new unknown function $v = u - U$. As the result we obtain the equivalent system of two first order equations

$$\frac{dv}{dX}=p,\quad \frac{dp}{dX}=\frac{2\alpha p-2\varepsilon p-v^2+U^2}{2\beta v} \tag{2.2}$$

Solutions of Eqs.(2.2) are symmetric relative to the vertical axis $v = 0$ of the phase plane, which is due to the appearance in the denominator of the right-hand side of the second equation of factor v and is the consequence of the invariance properties of the third order input equation (1.1) relative to the geometric group of Poincaré transforms /1/. The term v^2 in the numerator of the right-hand side of these same equation is related to the presence in (1.1) of the nonlinear term uu_x. When $\alpha = 0$ the pattern of integral curves is also symmetric about the horizontal axis $p = 0$ owing to the use in (1.1) of the specific form of the term $\varepsilon(u_x^2)_x$; when $\alpha \neq 0$ this symmetry is violated.

The right-hand sides of equations of system (2.2) imply that this system has two pairs of singular points whose coordinates are determined by conditions

$$p_{1,2}=0,\ v_{1,2}=\pm U \tag{2.3}$$

$$v_{3,4}=0,\ p_{3,4}=(\alpha\mp\sqrt{\alpha^2+2\varepsilon U^2})(2\varepsilon)^{-1} \tag{2.4}$$

Singularities with coordinates (2.3) reduce the right-hand sides of both equations (2.2) to zero. In the space (v, p, X) the integral curves may pass through these singular points when $X\to\infty$ or $X\to-\infty$.

Singularities with coordinates (2.4) occur only in the second equation of system (2.2), while the right-hand side of the first equation remains finite in the neigborhood of these singularities. Because of this there are two straight lines in space (v, p, X) which are parallel to the X-axis. Each point of these straights are singular points of equations of system (2.2). Thus the integral curves may pass in space (v, p, X) through singular points (2.4) for any finite X.

Let us investigate the dependence of properties of singular points with coordinates (2.3) and (2.4) on coefficients α, β, ε, U in the right-hand sides of Eqs.(2.2). Let us, first, consider singularities appearing on the v-axis when conditions (2.3) are satisfied. For this we linearize the expressions in the right-hand sides of Eqs.(2.2) in the neighborhood of singular points with coordinates (2.3), construct the respective secular equation, and determine its roots

$$\lambda_{1,2}=(\alpha\pm\sqrt{\alpha^2-4\beta U^2})(2\beta v)^{-1} \tag{2.5}$$

The roots $\lambda_{1,2}$ must be calculated for $v=\pm U$ for each region respectively. It follows from (2.5) that for $\beta > 0$, $\alpha^2 > 4\beta U^2$ each point with coordinates $v = +U, p = 0$ (subsequently denoted by the numeral *1*) represents a node with positive characteristic direction ($\lambda_1 > \lambda_2 > 0$), and the singular point with coordinates $v = -U, p = 0$ (denoted below by the numeral *2*) is a node with negative characteristic directions ($\lambda_1 < \lambda_2 < 0$); when $\alpha^2 \geqslant 4\beta U^2$ we have in *1* and *2* degenerated nodes with positive and negative characteristic directions, respectively; when $\alpha^2 < 4\beta U^2$ both singularities are of the focus type, and when $\alpha = 0$ both singularities are of the center type. When $\beta < 0$ the singular points *1* and *2* are saddles for any α and U, since $\alpha < \sqrt{\alpha^2 - 4\beta U^2}$.

The solution of equations of system (2.2) in the neighborhood of singularities (2.3) implies that when singularities *1* and *2* are nodes, foci or centers, $X \to -\infty$ corresponds to singular point *1*, and $X \to +\infty$ to singular point *2*. When singularities *1* and *2* are saddle points, there are two integral curves (one with positive and the other with negative characteristic direction) which pass through point *1*, and two similar ones that pass through point *2*. In the space (v, p, X) it is possible to move out from point *1*, along the integral curve $\lambda_1 > 0$, which shows that to that integral curve at point *1* corresponds $X \to -\infty$, and along the second integral curve it is possible to enter point *1*, i.e. to it corresponds $X \to +\infty$. The situation at point *2* is similar.

Let us investigate the properties of singularities on the p-axis, which occur when conditions (2.4) are satisfied. We denote the point with coordinates $v = 0$, $p = p_3$ by the numeral *3* and that with coordinates $v = 0, p = p_4$ by the numeral *4*. To do this we linearize the denominator and numerator in the right-hand side of the second equation in the neighborhood of singularities *3* and *4*, using the expression $\Delta v = p\Delta X$ for the increments of v. We obtain the equation

$$\frac{d\Delta p}{d\Delta X} = \frac{(2\alpha - 4\varepsilon p)\,\Delta p}{2\beta p \Delta X} \tag{2.6}$$

for which we write a secular equation whose roots are

$$\lambda_1 = 2\beta p, \ \lambda_2 = 2\alpha - 4\varepsilon p \tag{2.7}$$

Substituting in (2.7) the value of coordinate p_3 at point *3* and p_4 at point *4*, we find that when $\beta > 0$ both singularities the properties of a saddle, and at point *3* and *4* $\lambda_1 < 0$, $\lambda_2 > 0$ and $\lambda_1 > 0$, $\lambda_2 < 0$, respectively; when $\beta < 0$ we have at point *3* a singularity of the node type with $\lambda_1 > 0$, $\lambda_2 > 0$, and at point *4* a singularity of the node type with $\lambda_1 < 0$, $\lambda_2 < 0$.

When $\varepsilon = 0$ there remains only one singular point *3* with coordinate $p = -U^2/(2\alpha)$, which for $\beta > 0$ is a saddle with $\lambda_1 < 0$, $\lambda_2 > 0$, and when $\beta < 0$ a saddle with $\lambda_1 > 0, \lambda_2 > 0$. If with $\varepsilon = 0$ also $\alpha = 0$, i.e. the dissipative effects are absent, hence there are no singular points (2.4) on the p-axis.

Note also that when $\alpha = 0$, Eqs. (2.2) are integrable in quadratures. The general solution of (2.2) can be written in the form (when $\varepsilon \neq -\beta$ and $v \neq 0$)

$$2\varepsilon p^2 = U^2 + Cv^{-2\varepsilon/\beta} - v^2\varepsilon/(\varepsilon + \beta) \tag{2.8}$$

where C is the arbitrary constant of integration.

We would point out that the analysis of properties of singular points (2.4) was carried out in the plane (p, X). In the projection on the plane (v, p) the type of singular points is retained, and the characteristic directions prove to be parallel to the p- and v-axes.

The presence of singular points (2.4) in the considered system of equations is related to that Eq. (1.1) has a single characteristic velocity which in the moving coordinate system is $v = u - U$. Where that characteristic velocity vanishes, the steady solutions of the equation have singular points similar to those occurring in hyperbolic systems of equations and in equations with parabolic degeneration. Such singularities were investigated in detail in /11–13/ in the case of hyperbolic systems of general form and of a number of specific problems.

3. Let us show the qualitative patterns of integral curve behavior in the phase plane for several of characteristic cases considered above. The respective numerical steady solutions of Eq. (1.1) were obtained, using the implicit divergent difference scheme in which approximation of derivatives with respect to x in each secular layer was effected using symmetric differences on a five point pattern /1/.

Consider the problem of evolution of the front when initial data are specified by the distribution

$$u(x, 0) = [1 - \operatorname{th}(kx)]/2 \tag{3.1}$$

where $k = \text{const}$ defines the initial front curvature. Solution of the quasi-linear first order equation (1.2) with initial data (3.1) assumes in the course of time a steady discontinuous profile representing a jump of function u from 0 to 1 and moving at velocity $U = 0.5$ in the direction of increasing x. Solution of the Burgers equation (1.3) with input data (3.1) also assumes a steady profile in the form of a monotonic function. In the phase plane of variables $v = u - U$ and $p = dv/dX$ a section of parabola $p = (v^2 - U^2)/(2\alpha)$ for $p \leqslant 0$ corresponds to that solution.

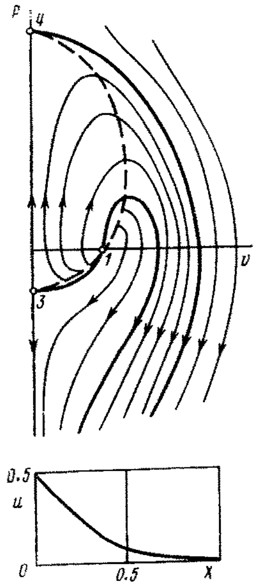

Fig.1

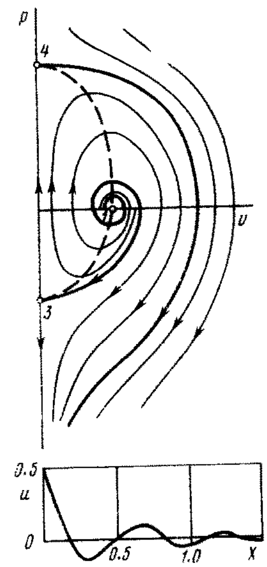

Fig.2

A qualitative pattern of integral curve behavior in the (v, p) phase plane is shown in Fig.1 for the quasi-linear third order equation (1.1) when $\beta > 0$ and $\alpha^2 > 4\beta U^2$. The separatrices that pass through singular saddle points are shown there by heavy lines, and the dash line represent the isocline of zero angles of inclination of integral curves defined by the equation $v^2 + 2\varepsilon p^2 - 2\alpha p = U^2$ of the ellipse. In this case we have a monotonic steady solution in the form of the front transition from $u = 1$ to $u = 0$ moving at velocity $U = 0.5$ which is the same as that of the discontinuous solution of Eq.(1.2). The section of the separatrix of saddle 3 which connects the singular nodal points 1 and 2 corresponds to that solution in the phase plane. Owing to the symmetry of integral curves relative to the ordinate axis only region $v \geqslant 0$ is shown in Fig.1 and subsequent. The dependence of steady distribution of u on coordinate X, obtained by numerical integration of (1.1) for coefficients $\beta = \varepsilon = 0.01$ and $\alpha = 0.1$ and input data (3.1) is also shown in Fig.1, where (and subsequently) function $u = u(X)$ is only constructed for $X \geqslant 0$.

The integral curves shown in Fig.2 relate to the case of $\beta > 0$ and $\alpha < 4\beta U^2$. All notation in this and subsequent figures conforms to that in Fig.1. In this case the steady solution in the transition zone of the shock wave is not monotonic and contains symmetric oscillations that are damped with increasing distance from the basic front. The integral curve which represents transition from $u = 1$ to $u = 0$ is a separatrix passing through the singular point 3. The respective steady numerical solution of Eq.(1.1) is also shown in Fig.2 for $\beta = \varepsilon = 0.01$ and $\alpha = 0.01$.

The integral curves and numerical solution for $\beta = \varepsilon = 0.01$ and $\alpha = 0$ are shown in Fig.3 for the case when singular points on the horizontal axis become centers. In that case the steady solution has undamped oscillations that propagate on X to infinity in both the positive and negative directions. When $X = 0$ we have a transition front stipulated in the input conditions. The motion along the separatrix of saddle 3 from point A to point B lying symmetrically relative to the coordinate origin, then, as $X \to +\infty$ the periodic motion from point B to point 4, from point 4 along another separatrix $v = 0$ to point 3, from point 3 to point 4 through point B, etc., corresponds to that front in the phase plane (v, p). When $X \to -\infty$ the motion is along the integral curve symmetric relative to the p-axis.

Let us now consider the case of $\beta < 0$ to which corresponds the field of integral curves shown in Fig.4. In this case the solution for input data (3.1) is defined by the integral curve issuing from saddle *1* and entering saddle *2* through node *3*. The corresponding numerical solution which is a monotonic function and defines the transition front from $u=1$ to $u=0$ is shown in Fig.4 for $\beta=-0.01, \varepsilon=|\beta|/2$ and $\alpha=0.01$.

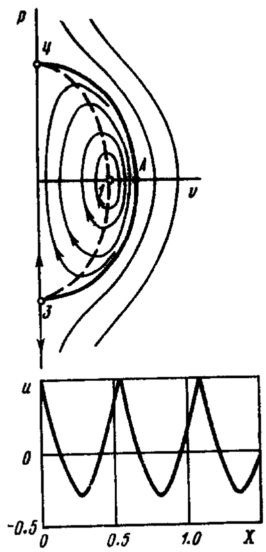

Fig.3

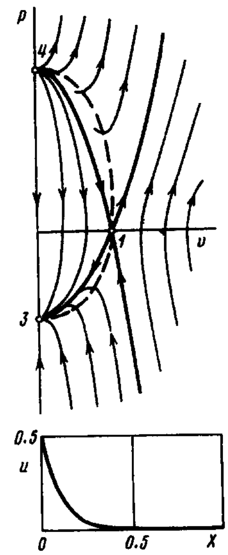

Fig.4

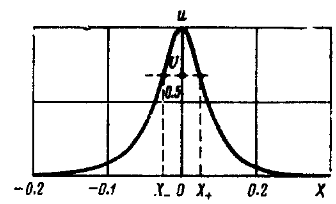

Fig.5

As $\varepsilon\to 0$ two singular points on the p-axis approach, respectively, $\mp\infty$, and the transition front length along the X-axis diminishes (the front slope increases). When coefficients α and β simultaneously approach zero, the considered above solutions of Eq.(1.1) become the corresponding discontinuous solution of Eq.(1.2). If then the condition $\alpha^2\geqslant 4\beta U^3$ remains valid, the obtained sequence of solutions is defined by monotonic functions, while in the opposite case we have a sequence of solutions with symmetrically damped ocillations.

Equation (1.1) has besides steady solutions of the type of shock wave transition zone when $\beta<0$ has steady solutions in the form of solitones, /1/. Such steady solution with amplitude equal unity propagating at the velocity $U=0.67$ calculated for $\alpha=0$ and $\beta=2\varepsilon=0.001$. In Fig.4 in the phase plane to that solution correspond the following sections of integral curves: the separatrix of saddle *2* which connects it to node *4*, then one of the integral curves that connect nodes *4* and *3* when $0<v<U$ and, finally the separatrix of saddle *2* which passes through node *3* and saddle *2*. To these three sections of integral curves correspond the following values of X: $-\infty<X\leqslant X_-$, $X_-\leqslant X\leqslant X_+$, $X_+\leqslant X<\infty$ (Fig.5). Due to the presence in the phase plane of singular nodal points *3* and *4*, Eq.(1.1) has an infinite number of various solitone solutions moving at the same velocity U when $\beta<0$.

We note in concluding that the steady solutions of equations investigated here substantially differ from those of the Korteweg—de Vries—Burgers equation /14,15/, as well as from the regularized equation for long waves /3/ whose steady solution structure is similar to that of the Korteweg—de Vries equation.

The authors thank A. G. Kulikovskii for valuable advice and interest in this work.

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Translated by J.J.D.